Journal of Computational and Applied Mathematics 233 (2009) 188–196

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

Existence of positive solutions of nonlinear m-point BVP for an increasing homeomorphism and positive homomorphism on time scales[☆]

Wei Han[a,*], Zhen Jin[a], Shugui Kang[b]

[a] Department of Mathematics, North University of China, Taiyuan Shanxi, 030051, PR China

[b] College of Mathematics and Computer Science, Shanxi Datong University, Datong, Shanxi 037009, PR China

ARTICLE INFO

Article history:
Received 17 January 2009
Received in revised form 4 July 2009

Keywords:
Time scale
Positive solutions
Boundary value problem
Fixed point theorems
Increasing homeomorphism and positive homomorphism

ABSTRACT

In this paper, we will consider the following multipoint boundary value problem for the following second-order dynamic equations on time scales

$$(\phi(u^{\Delta}))^{\nabla} + a(t)f(t, u(t)) = 0, \quad t \in (0, T),$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad \phi(u^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i \phi(u^{\Delta}(\xi_i)),$$

where $\phi : R \longrightarrow R$ is an increasing homeomorphism and positive homomorphism and $\phi(0) = 0$. By using fixed point theorems, we obtain an existence theorem of positive solutions for the above boundary value problem, which includes and improve some related results in the relevant literature. As an application, an example to demonstrate our results is given.

1. Introduction

A time scale $\mathbf{T}$ is a nonempty closed subset of R. We make the blanket assumption that $0, T$ are points in $\mathbf{T}$. By an interval $(0, T)$, we always mean the intersection of the real interval $(0, T)$ with the given time scale, that is $(0, T) \cap \mathbf{T}$.

In this paper, we will be concerned with the existence of positive solutions for the following dynamic equations on time scales:

$$(\phi(u^{\Delta}))^{\nabla} + a(t)f(t, u(t)) = 0, \quad t \in (0, T), \tag{1.1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad \phi(u^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i \phi(u^{\Delta}(\xi_i)), \tag{1.2}$$

where $\phi : R \longrightarrow R$ is an increasing homeomorphism and positive homomorphism and $\phi(0) = 0$.

A projection $\phi : R \longrightarrow R$ is called an increasing homeomorphism and positive homomorphism, if the following conditions are satisfied:

☆ The first author was supported by the Youth Science Foundation of North University of China. The second author was supported by the National Natural Science Foundation of China under Grant No 60771026, the Programme for New Century Excellent Talents in University (NCET050271), and the Special Scientific Research Foundation for the Subjects of Doctors in University (20060110005). The third author was supported by Natural Science Foundation of Shanxi Province (2008011002-1) and by Development Foundation of Higher Education Department of Shanxi Province.

* Corresponding author.
E-mail address: sh_hanweiwei1@126.com (W. Han).

0377-0427/$ – see front matter © 2009 Elsevier B.V. All rights reserved.
doi:10.1016/j.cam.2009.07.009

(i) if $x \le y$, then $\phi(x) \le \phi(y)$, $\forall\, x, y \in R$;
(ii) ϕ is a continuous bijection and its inverse mapping is also continuous;
(iii) $\phi(xy) = \phi(x)\phi(y)$, $\forall\, x, y \in R_+ = [0, +\infty)$.

We will assume that the following conditions are satisfied throughout this paper:

(H_1) $0 < \xi_1 < \cdots < \xi_{m-2} < \rho(T)$, $a_i,\ b_i \in [0, +\infty)$ satisfy $0 < \sum_{i=1}^{m-2} a_i < 1$, and $\sum_{i=1}^{m-2} b_i < 1$;
(H_2) $a(t) \in C_{ld}((0, T), [0, +\infty))$ and there exists $t_0 \in (\xi_{m-2}, T)$, such that $a(t_0) > 0$;
(H_3) $f \in C([0, T] \times [0, +\infty),\ [0, +\infty))$. (The Δ-derivative and the ∇-derivative in (1.1), (1.2) and the C_{ld} space in (H_2) are defined in Section 2.)

For the existence problems of positive solutions of boundary value problems on time scales, some authors have obtained many results in recent years, see [1–6,27] and the references therein. Recently, much attention has been paid to the existence of positive solutions for second-order nonlinear boundary value problems on time scales, for examples, see [7–9,6,10,29] and the references therein. At the same time, multipoint nonlinear boundary value problems with p-Laplacian operators on time scales have also been studied extensively in the literature, for details, see [1,3,4,11,12,6,28] and the references therein. But to the best of our knowledge, few people considered the second-order dynamic equations of increasing homeomorphism and positive homomorphism on time scales.

The present work is motivated by the recent papers [13–15]. In [15], Yang and Xiao studied the existence of multiple positive solutions for the following multipoint BVP:

$$(\phi(x'(t)))' + q(t)f(t, x(t), x'(t)) = 0, \quad t \in (0, 1), \tag{1.3}$$

$$x(0) = \sum_{i=1}^{n-2} \alpha_i x(\xi_i), \qquad \phi(x'(1)) = \sum_{i=1}^{n-2} \beta_i \phi(x'(\xi_i)), \tag{1.4}$$

where $\phi : R \longrightarrow R$ is an odd, increasing homeomorphism from R to R. By using the fixed point theorems, they obtained new results on the existence of at least three positive solutions of the above boundary value problem.

The rest of the paper is arranged as follows. We state some basic time scale definitions and prove several preliminary results in Section 2, Section 3 is devoted to establish the key conditions in Theorem 3.1 to show the existence of positive solutions of the BVP (1.1) and (1.2), the main tool being a recent new fixed point theorem in cone [6,10]. At the end of the paper, we will give two examples which illustrate that our work is true. We also point out that when $\mathbf{T} = R$, $p = 2$, (1.1) and (1.2) becomes a boundary value problem of differential equations and is just the problem considered in [16]. Our main results include and extend the main results of [13,14,16,17].

2. Preliminaries and some lemmas

For convenience, we list the following definitions which can be found in [18–21].

Definition 2.1. A time scale $\mathbf{T}$ is a nonempty closed subset of real numbers R. For $t < \sup \mathbf{T}$ and $r > \inf \mathbf{T}$, define the forward jump operator σ and backward jump operator ρ, respectively, by

$$\sigma(t) = \inf\{\tau \in \mathbf{T} \mid \tau > t\} \in \mathbf{T},$$
$$\rho(r) = \sup\{\tau \in \mathbf{T} \mid \tau < r\} \in \mathbf{T}$$

for all $t, r \in \mathbf{T}$. If $\sigma(t) > t$, t is said to be right scattered, and if $\rho(r) < r$, r is said to be left scattered; if $\sigma(t) = t$, t is said to be right dense, and if $\rho(r) = r$, r is said to be left dense. If $\mathbf{T}$ has a right scattered minimum m, define $\mathbf{T}_k = \mathbf{T} - \{m\}$; otherwise set $\mathbf{T}_k = \mathbf{T}$. If $\mathbf{T}$ has a left scattered maximum M, define $\mathbf{T}^k = \mathbf{T} - \{M\}$; otherwise set $\mathbf{T}^k = \mathbf{T}$.

Definition 2.2. For $f : \mathbf{T} \to R$ and $t \in \mathbf{T}^k$, the delta derivative of f at the point t is defined to be the number $f^{\Delta}(t)$, (provided it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)| \le \epsilon|\sigma(t) - s|,$$

for all $s \in U$.

For $f : \mathbf{T} \to R$ and $t \in \mathbf{T}_k$, the nabla derivative of f at t is the number $f^{\nabla}(t)$, (provided it exists), with the property that for each $\epsilon > 0$, there is a neighborhood U of t such that

$$|f(\rho(t)) - f(s) - f^{\nabla}(t)(\rho(t) - s)| \le \epsilon|\rho(t) - s|,$$

for all $s \in U$.

Definition 2.3. A function f is left-dense continuous (i.e. ld-continuous), if f is continuous at each left-dense point in $\mathbf{T}$ and its right-sided limit exists at each right-dense point in $\mathbf{T}$.

Definition 2.4. If $G^{\Delta} = f(t)$, then we define the delta integral by

$$\int_a^b f(t)\Delta t = G(b) - G(a).$$

If $F^{\nabla}(t) = f(t)$, then we define the nabla integral by

$$\int_a^b f(t)\nabla t = F(b) - F(a).$$

To prove the main results in this paper, we will employ several lemmas. especially Lemma 2.1 is based on the following linear BVP

$$(\phi(u^{\Delta}))^{\nabla} + h(t) = 0, \quad t \in (0, T), \tag{2.1}$$

$$u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \qquad \phi(u^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i \phi(u^{\Delta}(\xi_i)). \tag{2.2}$$

Lemma 2.1 *(See [22]). If $\sum_{i=1}^{m-2} a_i \neq 1$ and $\sum_{i=1}^{m-2} b_i \neq 1$, then for $h \in C_{ld}[0, T]$ the BVP (2.1) and (2.2) have the unique solution*

$$u(t) = \int_0^t \phi^{-1}\left(\int_s^T h(\tau)\nabla\tau - A\right)\Delta s + B, \tag{2.3}$$

where

$$A = -\frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T h(\tau)\nabla\tau}{1 - \sum_{i=1}^{m-2} b_i}, \qquad B = \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi^{-1}\left(\int_s^T h(\tau)\nabla\tau - A\right)\Delta s}{1 - \sum_{i=1}^{m-2} a_i}.$$

Let

$$E = \left\{u \in C_{ld}([0, T], R) \mid u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i), \phi(u^{\Delta}(T)) = \sum_{i=1}^{m-2} b_i \phi(u^{\Delta}(\xi_i))\right\},$$

where $C_{ld}([0, T], R)$ is the set of all ld-continuous functions from $[0, T]$ to R, and let the norm on E be the maximum norm. Then $(E, \|\cdot\|)$ is a Banach space. We define two cones by

$$P = \{u : u \in E, u(t) \geq 0, t \in [0, T]\},$$

and

$$P' = \{u : u \in E, u(t) \text{ is concave, nonnegative and increasing on } [0, T]\}.$$

Lemma 2.2. *If $u \in P'$, then for $t \in [0, T]$, we have $\inf_{t\in[0,T]} u(t) \geq \gamma \parallel u \parallel$, where*

$$\gamma = \frac{\sum_{i=1}^{m-2} a_i\xi_i}{\left(1 - \sum_{i=1}^{m-2} a_i\right)T + \sum_{i=1}^{m-2} a_i\xi_i}, \qquad \|u\| = \max_{t\in[0,T]} |u(t)|.$$

Proof. If $u \in P'$, then $u(t)$ is concave down on $(0, T)$.

For each $i \in \{1, 2, \ldots, m-2\}$, we have

$$\frac{u(T) - u(0)}{T - 0} \geq \frac{u(T) - u(\xi_i)}{T - \xi_i},$$

i.e.,

$$Tu(\xi_i) - \xi_i u(T) \geq (T - \xi_i)u(0),$$

so that

$$T\sum_{i=1}^{m-2} a_i u(\xi_i) - \sum_{i=1}^{m-2} a_i\xi_i u(T) \geq \sum_{i=1}^{m-2} a_i(T - \xi_i)u(0).$$

With the condition $u(0) = \sum_{i=1}^{m-2} a_i u(\xi_i)$, we have

$$u(0) \geq \frac{\sum\limits_{i=1}^{m-2} a_i\xi_i}{T - \sum\limits_{i=1}^{m-2} a_i(T-\xi_i)} u(T) = \frac{\sum\limits_{i=1}^{m-2} a_i\xi_i}{\left(1 - \sum\limits_{i=1}^{m-2} a_i\right) T + \sum\limits_{i=1}^{m-2} a_i\xi_i} u(T).$$

This completes the proof. □

Let

$$K = \{u | u \in E, u(t) \text{ is nonnegative and increasing on } [0, T], u(t) \geq \gamma \parallel u \parallel,\ t \in [0, T]\},$$

where γ is the same as in Lemma 2.2.

From Lemma 2.1, it is easy to see that the BVP (1.1) and (1.2) has a solution $u = u(t)$ if and only if u solves the equation

$$u(t) = \int_0^t \phi^{-1}\left(\int_s^T a(\tau)f(\tau, u(\tau))\nabla\tau - \tilde{A}\right)\Delta s + \tilde{B},$$

where

$$\tilde{A} = -\frac{\sum\limits_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)f(\tau, u(\tau))\nabla\tau}{1 - \sum\limits_{i=1}^{m-2} b_i}, \tag{2.4}$$

$$\tilde{B} = \frac{\sum\limits_{i=1}^{m-2} a_i \int_0^{\xi_i} \phi^{-1}\left(\int_s^T a(\tau)f(\tau, u(\tau))\nabla\tau - \tilde{A}\right)\Delta s}{1 - \sum\limits_{i=1}^{m-2} a_i}. \tag{2.5}$$

We define the operator $F : P \to E$ as follows

$$(Fu)(t) = \int_0^t \phi^{-1}\left(\int_s^T a(\tau)f(\tau, u(\tau))\nabla\tau - \tilde{A}\right)\Delta s + \tilde{B},$$

where $\tilde{A}$ and $\tilde{B}$ are defined by (2.4) and (2.5).

It is obvious that K is a cone in E. From the expression of $(Fu)(t)$, we can easily get $(Fu)(t) \in P'$, by Lemma 2.2, $F(K) \subset K$. So by applying the Arzela–Ascoli theorem on time scales [23], we can obtain that $F(K)$ is relatively compact. In view of Lebesgue's dominated convergence theorem on time scales [24], it is easy to prove that F is continuous. Hence, $F : K \to K$ is completely continuous.

The approach is mainly based on the following fixed point theorem, they can be found in [25,26].

Let E be a real Banach space and P be a cone in E. $\rho : P \longrightarrow R$ is said to be a convex functional on P if $\rho(tx + (1-t)y) \leq t\rho(x) + (1-t)\rho(y)$ for all $x, y \in P$ and $t \in [0, 1]$.

Lemma 2.3 (*See [25,26]*). *Let Ω_1 and Ω_2 be two bounded open sets in E such that $\theta \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Suppose that $A : P\bigcap(\overline{\Omega_2} \setminus \Omega_1) \longrightarrow P$ is completely continuous and $\rho : P \longrightarrow [0, +\infty)$ is a uniformly continuous convex functional with $\rho(\theta) = 0$ and $\rho(x) > 0$ for $x \neq \theta$. If one of the two conditions is satisfied:*

(H_4) $\rho(Ax) \leq \rho(x), \forall x \in P\bigcap \partial\Omega_1$ *and* $\inf_{x\in P\bigcap\partial\Omega_2} \rho(x) > 0$, $\rho(Ax) \geq \rho(x), \forall x \in P\bigcap\partial\Omega_2$,
(H_5) $\inf_{x\in P\bigcap\partial\Omega_1} \rho(x) > 0$, $\rho(Ax) \geq \rho(x), \forall x \in P\bigcap\partial\Omega_1$ *and* $\rho(Ax) \leq \rho(x), \forall x \in P\bigcap\partial\Omega_2$,

then A has at least one fixed point in $P\bigcap(\overline{\Omega_2} \setminus \Omega_1)$.

Remark 2.1 (*See [25,26]*). Obviously, $\rho(x) = \|x\|$ is a uniformly continuous convex functional with $\rho(\theta) = 0$ and $\rho(x) > 0$ for $x \neq \theta$. Moreover $\inf_{x\in P\bigcap\partial\Omega_1} \|x\| > 0$ and $\inf_{x\in P\bigcap\partial\Omega_2} \|x\| > 0$ since $\theta \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$.

To employ the above lemmas, we define $\rho(u) = \max_{t\in[0,T]} u(t) = \|u\|$, then $\rho : K \longrightarrow [0, +\infty)$ is a uniformly continuous convex functional with $\rho(\theta) = 0$ and $\rho(u) > 0$ for $u \neq 0$.

Now, for convenience, we introduce the following notations. Let

$$\varphi(s)=\phi^{-1}\left(\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau-\tilde{A}\right),$$

$$m = \left\{\int_0^T \phi^{-1}\left[\int_s^T a(\tau)\nabla\tau+\frac{\sum_{i=1}^{m-2} b_i\int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right]\Delta s\right.$$
$$\left.+\frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^{\xi_i}\phi^{-1}\left[\int_s^T a(\tau)\nabla\tau+\frac{\sum_{i=1}^{m-2} b_i\int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right]\Delta s\right\}^{-1}. \tag{2.6}$$

3. Existence theorems of positive solutions

Theorem 3.1. *Assume that* (H_1), (H_2) *and* (H_3) *hold, and if there exist constants a and b with* $0<b<a$ *satisfying one of the two conditions:*

(H_6) $a\geq\frac{b}{\gamma^2}$, $f(t,u)\geq\phi(am)$, *for* $t\in[0,T]$, $a\gamma\leq u\leq a$ *and* $f(t,u)\leq\phi(bm)$ *for* $t\in[0,T]$, $u\leq\frac{b}{\gamma}$,

(H_7) $f(t,u)\leq\phi(am)$, *for* $t\in[0,T]$, $u\leq\frac{a}{\gamma}$ *and* $f(t,u)\geq\phi(bm)$, *for* $t\in[0,T]$, $b\gamma\leq u\leq b$,

then (1.1) and (1.2) have at least a positive solution.

Proof. Set

$$\Omega_1=\{u\in C_{ld}([0,T],R)|\rho(u)<b\},\qquad \Omega_2=\{u\in C_{ld}([0,T],R)|\rho(u)<a\}.$$

It is clear that Ω_1 and Ω_2 are open sets with $\theta\in\Omega_1$ and $\overline{\Omega_1}\subset\Omega_2$.

Suppose (H_6) is satisfied. If $u\in K\bigcap\partial\Omega_1$, then $\rho(u)=b$ and for $t\in[0,T]$, $\|u\|\leq\frac{b}{\gamma}$. Therefore,

$$\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau-\tilde{A}=\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau+\frac{\sum_{i=1}^{m-2} b_i\int_{\xi_i}^T a(\tau)f(\tau,u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}$$
$$\leq\phi(bm)\left[\int_s^T a(\tau)\nabla\tau+\frac{\sum_{i=1}^{m-2} b_i\int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right],$$

so that

$$\varphi(s)=\phi^{-1}\left(\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau-\tilde{A}\right)$$
$$\leq bm\phi^{-1}\left[\int_s^T a(\tau)\nabla\tau+\frac{\sum_{i=1}^{m-2} b_i\int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right].$$

And thus, by (2.6), we have

$$\rho(Fu)=\max_{0\leq t\leq T}\left[\int_0^t\varphi(s)\Delta s+\tilde{B}\right]=\int_0^T\varphi(s)\Delta s+\tilde{B}$$
$$=\int_0^T\varphi(s)\Delta s+\frac{\sum_{i=1}^{m-2} a_i\int_0^{\xi_i}\varphi(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i}$$

$$
\begin{aligned}
&\leq bm\left\{\int_0^T \phi^{-1}\left[\int_s^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right]\Delta s\right.\\
&\quad\left. + \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^{\xi_i}\phi^{-1}\left[\int_s^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right]\Delta s\right\}\\
&= b = \rho(u).
\end{aligned}
$$

If $u \in K \bigcap \partial\Omega_2$, then $\rho(u) = a$ and for $t \in [0, T]$,

$$a\gamma = \gamma\|u\| \leq u(t) \leq \rho(u) = a.$$

Therefore,

$$
\begin{aligned}
\int_s^T a(\tau)f(\tau, u(\tau))\nabla\tau - \tilde{A} &= \int_s^T a(\tau)f(\tau, u(\tau))\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)f(\tau, u(\tau))\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\\
&\geq \phi(am)\left[\int_s^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right],
\end{aligned}
$$

so that

$$
\begin{aligned}
\varphi(s) &= \phi^{-1}\left(\int_s^T a(\tau)f(\tau, u(\tau))\nabla\tau - \tilde{A}\right)\\
&\geq am\phi^{-1}\left[\int_s^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right].
\end{aligned}
$$

And thus, it follows that

$$
\begin{aligned}
\rho(Fu) &= \max_{0\leq t\leq T}\left[\int_0^t \varphi(s)\Delta s + \tilde{B}\right] = \int_0^T \varphi(s)\Delta s + \tilde{B}\\
&= \int_0^T \varphi(s)\Delta s + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}\varphi(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i}\\
&\geq am\left\{\int_0^T \phi^{-1}\left[\int_s^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right]\Delta s\right.\\
&\quad\left. + \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^{\xi_i}\phi^{-1}\left[\int_s^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right]\Delta s\right\}\\
&= a = \rho(u).
\end{aligned}
$$

Thus, by Lemma 2.3, condition (H_4) is satisfied, (1.1) and (1.2) have at least one positive solution.

In the following, when (H_7) is satisfied, we check condition (H_5) in Lemma 2.3. If $u \in K \bigcap \partial\Omega_1$, then $\rho(u) = b$ and for $t \in [0, T]$,

$$b\gamma = \gamma\|u\| \le u(t) \le \rho(u) = a.$$

Therefore,

$$\begin{aligned}
\rho(Fu) &= \max_{0\le t\le T}\left[\int_0^t \phi^{-1}\left(\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau - \tilde{A}\right)\Delta s + \tilde{B}\right] \\
&= \int_0^T \phi^{-1}\left(\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau - \tilde{A}\right)\Delta s + \tilde{B} \\
&= \int_0^T \varphi(s)\Delta s + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}\varphi(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \\
&\ge bm\left\{\int_0^T \phi^{-1}\left[\int_s^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right]\Delta s \right.\\
&\quad \left. + \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^{\xi_i}\phi^{-1}\left[\int_s^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right]\Delta s\right\} \\
&= b = \rho(u).
\end{aligned}$$

If $u \in K \bigcap \partial\Omega_2$, then $\rho(u) = a$ and for $t \in [0, T]$, we have

$$\|u\| \le \frac{a}{\gamma}.$$

Therefore,

$$\begin{aligned}
\rho(Fu) &= \max_{0\le t\le T}\left[\int_0^t \phi^{-1}\left(\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau - \tilde{A}\right)\Delta s + \tilde{B}\right] \\
&= \int_0^T \phi^{-1}\left(\int_s^T a(\tau)f(\tau,u(\tau))\nabla\tau - \tilde{A}\right)\Delta s + \tilde{B} \\
&= \int_0^T \varphi(s)\Delta s + \frac{\sum_{i=1}^{m-2} a_i \int_0^{\xi_i}\varphi(s)\Delta s}{1-\sum_{i=1}^{m-2} a_i} \\
&\le am\left\{\int_0^T \phi^{-1}\left[\int_s^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right]\Delta s \right.\\
&\quad \left. + \frac{\sum_{i=1}^{m-2} a_i}{1-\sum_{i=1}^{m-2} a_i}\int_0^{\xi_i}\phi^{-1}\left[\int_s^T a(\tau)\nabla\tau + \frac{\sum_{i=1}^{m-2} b_i \int_{\xi_i}^T a(\tau)\nabla\tau}{1-\sum_{i=1}^{m-2} b_i}\right]\Delta s\right\} \\
&= a = \rho(u).
\end{aligned}$$

Thus, by Lemma 2.3, condition (H_5) is satisfied, so (1.1) and (1.2) have at least one positive solution. □

4. Some examples

In this section, we present a simple example to explain our results.

Example 4.1. Let $\mathbf{T} = \{(\frac{1}{2})^n : n \in N\} \bigcup \{1\}$, $T = 1$. Consider the following BVP on time scales

$$(\phi(u^{\Delta}))^{\nabla} + f(t, u(t)) = 0, \quad t \in (0, T), \tag{4.1}$$

$$u(0) = \frac{1}{4} u\left(\frac{1}{3}\right), \qquad \phi(u^{\Delta}(T)) = \frac{1}{2}\phi\left(u^{\Delta}\left(\frac{1}{3}\right)\right), \tag{4.2}$$

where

$$\phi(u) = \begin{cases} u^3, & u \le 0, \\ u, & u > 0, \end{cases} \quad f(t,u) = (1+t)\left(\frac{u}{20}\right)^{20}, \ (t,u) \in [0,1] \times [0, +\infty).$$

It is easy to check that $f : [0,1] \times [0,+\infty) \longrightarrow [0, +\infty)$ is continuous. In this case, $a(t) \equiv 1$, $a_1 = \frac{1}{4}$, $b_1 = \frac{1}{2}$, $\xi_1 = \frac{1}{3}$, it follows from a direct calculation that

$$\begin{aligned} m &= \left\{ \int_0^T \phi^{-1}\left[(T-s) + \frac{b_1(T-\xi_1)}{1-b_1}\right] \Delta s + \frac{a_1}{1-a_1} \int_0^{\xi_1} \phi^{-1}\left[(T-s) + \frac{b_1(T-\xi_1)}{1-b_1}\right] \Delta s \right\}^{-1} \\ &= \left[\int_0^1 \left(1 - s + \frac{\frac{1}{2}(1-\frac{1}{3})}{1-\frac{1}{2}}\right)^{\frac{1}{2}} \mathrm{d}s + \frac{\frac{1}{4}}{1-\frac{1}{4}} \int_0^{\frac{1}{3}} \left(1 - s + \frac{\frac{1}{2}(1-\frac{1}{3})}{1-\frac{1}{2}}\right)^{\frac{1}{2}} \mathrm{d}s \right]^{-1} \\ &= \left[\int_0^1 \left(\frac{5}{3} - s\right)^{\frac{1}{2}} \mathrm{d}s + \frac{1}{3} \int_0^{\frac{1}{3}} \left(\frac{5}{3} - s\right)^{\frac{1}{2}} \mathrm{d}s \right]^{-1} \approx 0.8281, \end{aligned}$$

$$\gamma = \frac{a_1 \xi_1}{(1-a_1)T + a_1\xi_1} = \frac{\frac{1}{4} \cdot \frac{1}{3}}{\left(1 - \frac{1}{4}\right) \cdot 1 + \frac{1}{4} \cdot \frac{1}{3}} = \frac{1}{10}.$$

Choose $a = 400$, $b = 1$, it is easy to check that $400 = a \ge \frac{b}{\gamma^2} = 100$,

$$\phi(am) = \phi(400 \times 0.8281) = (400 \times 0.8281) = 331.24,$$
$$\phi(bm) = \phi(0.8281) = 0.8281,$$

and f satisfies that

$$f(t,u) \ge \left(\frac{u}{20}\right)^{20} \ge \left(\frac{40}{20}\right)^{20} = 2^{20} \ge 331.24 = \phi(am), \quad \text{for } 40 \le u \le 400,$$

$$f(t,u) \le 2 \times \left(\frac{u}{20}\right)^{20} \le 2 \times \left(\frac{10}{20}\right)^{20} = 2 \times \left(\frac{1}{2}\right)^{20} \le 0.6857 = \phi(bm), \quad \text{for } u \le 10,$$

thus, condition (H_6) in Theorem 3.1 is satisfied, the BVP (4.1) and (4.2) have at least one positive solution.

If we take

$$f(t,u) = (1+t) \cdot \frac{u}{5000} + 1, \quad (t,u) \in [0,1] \times [0, +\infty).$$

By computation, we can get

$$f(t,u) \le 2 \times \frac{4000}{5000} + 1 \le 331.24 = \phi(am), \quad \text{for } u \le 4000,$$

$$f(t,u) \ge \frac{\frac{1}{10}}{5000} + 1 \ge 0.8281 = \phi(bm), \quad \text{for } b\gamma = \frac{1}{10} \le u \le 1 = b,$$

thus, condition (H_7) in Theorem 3.1 is satisfied, the BVP (4.1) and (4.2) has also at least one positive solution.

References

[1] D.R. Anderson, R. Avery, J. Henderson, Existence of solutions for a one-dimensional p-Laplacian on time scales, J. Difference Equ. Appl. 10 (2004) 889–896.

[2] H.Y. Feng, W. Ge, M. Jiang, Multiple positive solutions for m-point boundary-value problems with a one-dimensional p-Laplacian, Nonlinear Anal. 68 (2008) 2269–2279.

[3] Z.M. He, Double positive solutions of three-point boundary value problems for p-Laplacian dynamic equations on time scales, J. Comput. Appl. Math. 182 (2005) 304–315.

[4] Z.M. He, Triple positive solutions of boundary value problems for p-Laplacian dynamic equations on time scales, J. Math. Anal. Appl. 321 (2006) 911–920.

[5] D. Ma, Z. Du, W. Ge, Existence and iteration of monotone positive solutions for multipoint boundary value problem with p-Laplacian operator, Comput. Math. Appl. 50 (2005) 729–739.
[6] H.R. Sun, W.T. Li, Multiple Positive solutions for p-Laplacian m-point boundary value problems on time scales, Appl. Math. Comupt. 182 (2006) 478–491.
[7] D.R. Anderson, Solutions to second-order three-point problems on time scales, J. Difference Equ. Appl. 8 (2002) 673–688.
[8] J.J. Dacunha, J.M. Davis, P.K. Singh, Existence results for singular three point boundary value problems on time scales, J. Math. Anal. Appl. 295 (2004) 378–391.
[9] J.P. Sun, W.T. Li, Existence and nonexistence of positive solutions for second-order time scale systems, Nonlinear Anal. 68 (2008) 3107–3114.
[10] H.R. Sun, W.T. Li, Positive solutions for nonlinear three-point boundary value problems on time scales, J. Math. Anal. Appl. 299 (2004) 508–524.
[11] H. Luo, Q.Z. Ma, Positive solutions to a generalized second-order three-point boundary value problem on time scales, Electron. J. Differential Equations 17 (2005) 1–14.
[12] H. Su, B. Wang, Z. Wei, Positive solutions of four-point boundary value problems for four-order p-Laplacian dynamic equations on time scales, Electron. J. Differential Equations 78 (2006) 1–13.
[13] S.H. Liang, J.H. Zhang, The existence of countably many positive solutions for nonlinear singular m-point boundary-value problems, J. Comput. Appl. Math. 214 (2008) 78–89.
[14] B.F. Liu, J.H. Zhang, The existence of positive solutions for some nonlinear boundary value problems with linear mixed boundary conditions, J. Math. Anal. Appl. 309 (2005) 505–516.
[15] Yu Yang, Dongmei Xiao, On existence of multiple positive solutions for ϕ-Laplacian multipoint boundary value, Nonlinear Anal. 71 (2009) 4158–4166.
[16] R.Y. Ma, Existence of solutions of nonlinear m-point boundary value problem, J. Math. Anal. Appl. 256 (2001) 556–567.
[17] Y. Wang, W. Ge, Positive solutions for multipoint boundary value problems with a one-dimensional p-Laplacian, Nonlinear Anal. 66 (2007) 1246–1256.
[18] R.P. Agarwal, D. O'Regan, Nonlinear boundary value problems on time scales, Nonlinear Anal. 44 (2001) 527–535.
[19] F.M. Atici, G.Sh. Gnseinov, On Green'n functions and positive solutions for boundary value problems on time scales, J. Comput. Anal. Math. 141 (2002) 75–99.
[20] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, Cambridge, MA, 2001.
[21] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, Cambridge, MA, 2003.
[22] Wei Han, Zhen Jin, Positive solutions of nonlinear m-point BVP for an increasing homeomorphism and positive homomorphism with sign changing nonlinearity on time scales, Commun. Nonlinear Sci. Numer. Simul. (Corrected Proof) (in press).
[23] R.P. Agarwal, M. Bohner, P. Rehak, Half-linear Dynamic equations, Nonlinear Analysis and Applications: To V. Lakshmikantham on his 80th Birthday, Kluwer Academic Publishers, Dordrecht, 2003, pp. 1–57.
[24] B. Aulbach, L. Neidhart, Integration on measure chain, in: Proc. of the Sixth Int. Conf. on Difference Equations, CRC, BocaRaton, Fl, 2004, pp. 239–252.
[25] Guowei Zhang, Jingxian Sun, A generalization of the cone expansion and compression fixed point theorem and applications, Nonlinear Anal. 67 (2007) 579–586.
[26] D. Guo, V. Lakshmikantham, Nonlinear Problems in Abstract Cones, Academic Press, San Diego, 1988.
[27] K.Q. Lan, Multiple positive solutions of semilinear differential equations with singularities, J. London Math. Soc. 63 (2001) 690–704.
[28] H. Su, Z. Wei, F. Xu, The existence of countably many positive solutions for a system of nonlinear singular boundary value problems with the p-Laplacian operator, J. Math. Anal. Appl. 325 (2007) 319–332.
[29] Positive solutions for nonlinear m-point boundary value problems on time scalesH.R. Sun, W.T. Li, Acta Math. Sinica 49 (2006) 369–380 (in Chinese).